

GRADUATE PRELIMINARY EXAMINATION

ANALYSIS

Sample C

1. Let f be a real valued function defined on a set D which is dense in $[0, 1]$. If f is uniformly continuous on D , show that f can be extended to a uniformly continuous function on $[0, 1]$.

2. Fix a real number $a > 1$ and define a sequence of numbers $\{x_n\}$ inductively by $x_1 = 0$ and

$$x_{n+1} = \frac{a(1+x_n)}{a+x_n} \text{ for } n = 0, 1, \dots$$

Show that $\lim_{n \rightarrow \infty} x_n$ exists and find this limit.

3. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a given function, and consider 3 variables x, y, z which are related by the equation

(A)
$$f(x, y, z) = 0 .$$

In some textbooks in Thermodynamics it is claimed that (A) implies the formula

(B)
$$(\partial x / \partial y)(\partial y / \partial z)(\partial z / \partial x) = -1 ,$$

where it is understood that equation (A) may be solved for each variable in terms of the other two. Thus, for example, x may be expressed as function of y and z , and it is this function which is differentiated in the symbol $\frac{\partial x}{\partial y}$. By making use of the Implicit Function Theorem, formulate and prove a precise theorem, including appropriate hypotheses on f , which shows that (B) is indeed a consequence of (A).

4. Let f be a continuous, nonnegative function defined on $[a, b]$ with $M = \sup_{x \in [a, b]} f(x)$. Prove

$$\lim_{n \rightarrow \infty} \left(\int_a^b f^n(x) dx \right)^{1/n} = M .$$

5. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(0, 0) = 0$, and $f(s, t) = t^3 / (s^2 + t^2)$ otherwise. Prove the following facts:

- (a) f is continuous on \mathbb{R}^2 ,
- (b) $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ exist at all points of \mathbb{R}^2 ,
- (c) f is not differentiable at $(0, 0)$.