GRADUATE PRELIMINARY EXAMINATION

ANALYSIS

Sample B

1. Let $\{a_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=0}^{\infty} a_n = 1$. The power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [-1,1]$. If L denotes the left-hand derivative of f at $x = 1$, $L = \lim_{x \to 1^-} \frac{f(1) - f(x)}{1 - x}$, show that

$$L = \sum_{n=1}^{\infty} n a_n ,$$

including the case $+\infty = +\infty$.

2. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $\mathbb{R}$ such that $f_n(0) = 0$ for all n, $f'_n \cdot f'_m \equiv 0$ for all $m \neq n$, and $f'_n \to 0$ uniformly on $\mathbb{R}$ as $n \to \infty$.
   (a) Prove that $\sum_1^{\infty} f'_n$ converges uniformly and absolutely on $\mathbb{R}$. Let $g = \sum_1^{\infty} f'_n$.
   (b) Prove that $\sum_1^{\infty} f_n$ converges pointwise on $\mathbb{R}$. Let $f = \sum_1^{\infty} f_n$.
   (c) Show that f is differentiable on $\mathbb{R}$, and that $f'(x) = g(x)$ for all $x \in \mathbb{R}$.

3. Let $g, f_n, n = 1,2,\ldots$ be real valued functions defined on $[0,\infty)$ such that: (i) each f_n is Riemann integrable on every interval $[0,T]$, $T < \infty$; (ii) $|f_n(x)| \leq g(x)$ for all n and x; (iii) $\int_0^{\infty} g(x)dx < \infty$, and (iv) there is a function f such that $f_n \to f$ uniformly on every interval $[0,T]$ as $n \to \infty$. Prove, without using results from Lebesgue integration theory, that the improper Riemann integrals $\int_0^{\infty} f_n(x)dx$ and $\int_0^{\infty} f(x)dx$ exist, and

$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)dx = \int_0^{\infty} f(x)dx .$$

4. Determine the convergence (absolute or conditional) or divergence of the following series:

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} n^2[\pi^{1/n} - 1]^n$

(c) $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}$ (d) $\sum_{n=1}^{\infty} n!\, e^{-n}$