GRADUATE PRELIMINARY EXAMINATION

ANALYSIS

Sample A

1. (a) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of real numbers. Show by example that the series $\sum_{n=1}^{\infty} a_n b_n$ need not converge.

(b) If, in addition, $a_n \geq 0$ and $b_n \geq 0$ for all n, prove that $\sum_{n=1}^{\infty} a_n b_n$ does converge.

2. Prove $x^{-1}\arctan(x)$ is decreasing on $[1, \infty)$.

3. (a) Prove that if f is a continuous, strictly positive function on $[0,1]$, then $\int_0^1 f(x)dx > 0$. You may assume only the definition of the integral.

(b) Prove the same thing if f is only assumed to be Riemann integrable and strictly positive on $[0,1]$. Here you may assume basic facts from analysis, other than what you are asked to prove. For instance, you may assume $\int_0^1 f(x)dx \geq 0$ for nonnegative Riemann integrable functions f.

4. (a) Suppose f is a continuous, real valued function defined on (a,b). Let $c \in (a,b)$, and suppose that f is differentiable on $E = (a,c) \cup (c,b)$ and $f'(x) \to \lambda$ as $x \to c$ in E. Prove that $f'(c)$ exists and equals λ.

(b) Let g be a continuous, real valued function on $[a,b]$. Suppose that g is differentiable on (a,b), and that $g'(x) \neq 0$ for all $x \in (a,b)$. Prove that $g^{-1}(y)$ is a finite set for all y in the range of g.

5. Is it possible to solve

$$xy^2 + xzu + yv^2 = 3$$
$$u^3yz + 2xv - u^2v^2 = 2$$

for $u(x,y,z)$ and $v(x,y,z)$ near $(x,y,z) = (1,1,1)$ such that $(u(1,1,1), v(1,1,1)) = (1,1)$? Why? If it is possible, compute $\frac{\partial v}{\partial y}$ at $(1,1,1)$.

6. Suppose f is differentiable on the interval $[a,b]$, $f(a) = 0$, and there is a finite constant A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$. Hint: For $a \leq c \leq b$ let $M_c = \sup\{|f(x)| : a \leq x \leq c\}$, and show that $|f(x)| \leq AM_c(x-a)$ for all $x \in [a.c]$.