Analysis Exam August 2007

1. Show that any set E in a connected metric space X with no boundary in X is either X or empty. Note: if we denote the closure of E by $\overline{E}$ and the complement of E by E^c then the boundary of E is given by $\overline{E} \cap \overline{E^c}$.

2. Suppose that a function f is defined on $[0, \infty)$, bounded on any interval $[0, a]$, $a < \infty$, and $\lim_{x\to\infty}(f(x+1) - f(x))$ exists. Show that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} (f(x+1) - f(x)).$$

3. Suppose that $\sum a_n$ and $\sum b_n$ are series with non-negative terms and the series $\sum b_n$ converges. Show that if

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for all $n \geq n_0$, then the series $\sum a_n$ also converges.
Derive that $\sum a_n$ converges if $a_n > 0$ and if there is a $p > 1$ so that $\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$ for all n. Hint: use $b_n = n^{-p}$.

4. Let $f(x)$ be continuous on $[0, 1]$ and suppose that

$$\int_0^1 f(x)\, x^n\, dx = \frac{1}{n+1}$$

for all $n = 0, 1, 2, \ldots$. What can you say about the function $f(x)$? Prove your answer.

5. Prove that the only function $f(x)$ satisfying $f^2(x)$ is Riemann Integrable on $[0, 1]$ and

$$f(x) = \int_0^x f^2(t)\, dt \text{ for } x \in [0, 1]$$

is the function $f(x) \equiv 0$.

6. Consider the map $(u, v) = \mathbf{f}(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $u = x^2 + y^2$, $v = x^2 + y^2 - y$.
(a) Find all the points (x, y) so that $\mathbf{f}(x, y) = (1, 1/2)$.
(b) Choose one of the points you found in (a) and call it $\mathbf{a} = (x_0, y_0)$. What does the inverse function theorem say about $\mathbf{f}$ near $\mathbf{a}$? State your answer carefully.
(c) Why is (a) not a contradiction to (b)?