Preliminary Examination in Analysis, January 2004

1. Show that if $E \subseteq \mathbb{R}^k$ is not compact then there is a continuous function $f : E \to \mathbb{R}$ which is unbounded.

2. Let $f : (0, +\infty) \to \mathbb{R}$ be a differentiable function such that $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = 0$. Prove that there is a sequence $x_n \nearrow +\infty$ such that $f'(x_n) \to 0$.

3. Let $f : [x_1, x_2] \to \mathbb{R}$ be a differentiable function, where $0 < x_1 < x_2$. Prove that there exists $c \in (x_1, x_2)$ such that

$$\frac{1}{x_1 - x_2} \begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(c) - cf'(c).$$

4. Let $f, \rho : [0, +\infty) \to \mathbb{R}$ be functions which are Riemann integrable on each interval $[0, A]$, $A > 0$. Assume that $\rho(x) \geq 0$ for all $x \geq 0$ and

$$\int_0^{+\infty} \rho(x)\, dx = 1, \quad \lim_{x \to +\infty} f(x) = L \in \mathbb{R}.$$

(i) Calculate $t \displaystyle\int_0^{+\infty} \rho(tx)\, dx$, where $t > 0$.

(ii) Show that $\lim\limits_{t \searrow 0} t \displaystyle\int_0^{+\infty} \rho(tx) f(x)\, dx = L$.

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n + x^{2n}}$. Find all the values $x \geq 0$ where the series is convergent. Show that the series converges uniformly on the set $[0, 1/2] \cup [2, +\infty)$. Is the series uniformly convergent on $[0, 1)$? Justify your answer.

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that f is uniformly continuous on $\{(x, y) : x^2 + y^2 \leq 1\}$. Find the first order partial derivatives of f at $(0, 0)$. Is f differentiable at $(0, 0)$? Justify your answer.