Analysis Preliminary Exam **August 16 2003**

1. If f is continuous on $[a,b]$ and

$$F(x) = \int_a^x f(t)\, dt$$

for $x \in [a,b]$, show that $F' = f$ on (a,b).

2. Prove that

$$\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln n \to \gamma$$

for some $\gamma \in (1/2, 1)$.

3. Let $f : \mathbf{R^1} \to \mathbf{R^1}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational or } x = 0 \\ p \sin \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with no common divisors.} \end{cases}$$

Where is f continuous?

4. For each n let $f_n : \mathbf{R^1} \to \mathbf{R^1}$ be a non-decreasing function, and assume f_n converges point-wise to a continuous function f. Prove that f_n converges uniformly on compact sets to f.

5. Let f be a continuous function on $[0,1]$ such that

$$\int_0^1 e^{-\frac{nx}{1-x}} f(x)\, dx = 0$$

for all $n \geq 0$. Show that f is identically zero.

6. Show that there is an open interval I containing 0 and a unique curve $(x(t), y(t)), t \in I$ with $(x(0), y(0)) = (1,1)$ satisfying

(*)
$$x + y^2 + \sin t = 2$$
$$x^2 + ty^2 = 1.$$

Find the velocity of the curve at $t = 0$. For a given $t_0 \in I$ is there a unique solution (x, y) to (*) with $t = t_0$?