Preliminary Exam - January 2002

1. Let A and B be subsets of a metric space. Prove that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and give an example when $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2. Let f and f' be continuous functions on $\mathbb{R}$. Prove that that the sequence of functions

$$g_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}$$

converges to $f'(x)$ uniformly on every interval $[a, b]$, $-\infty < a < b < \infty$.

3. Let f be a Riemann integrable function on $[0, 1]$ and

$$F(x) = \int_0^x f(t)\, dt.$$

a) Show that there is a constant C such that $|F(x) - F(y)| \leq C|x - y|$ for every $x, y \in [0, 1]$.

b) Give an example of f such that F is not differentiable at some point.

4. Show that the sequence

$$f_n(x) = \frac{\tan^{-1}(nx)}{\sqrt{n}}$$

is equicontinuous on $\mathbb{R}$ and converges uniformly to $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f_n'(x)$ does not converge uniformly to $f'(x)$.

5. Determine the values of α for which f is differentiable at $(0, 0)$ when

$$f(x, y) = \begin{cases} (x^2 + y^2)^\alpha \sin \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0); \\ 0, & (x, y) = (0, 0). \end{cases}$$

6. Show that if $\phi(y)$ is a continuously differentiable function on $(-a, a)$, $a > 0$, such that $\phi(0) = 0$ and $|\phi'(y)| \leq k < 1$ on $(-a, a)$, then there is $\varepsilon > 0$ and a unique differentiable function g on $(-\varepsilon, \varepsilon)$ satisfying the equation $x = g(x) + \phi(g(x))$.