Preliminary Examination

Analysis

18 August 1997

1. Let $K \subset \mathbb{R}^n$ be a compact set and let $\epsilon > 0$. Set $J = \{x \in \mathbb{R}^n \mid \text{dist}(x, K) \leq \epsilon\}$, where $\text{dist}(x, K) = \inf\{\|x - y\|_2 \mid y \in K\}$ and $\|t\|_2$ is the usual norm in $\mathbb{R}^n$. Prove that J is compact.

2. Determine the convergence or divergence of the following sequences $\{x_n\}_{n=1}^{\infty}$.

   (a) $x_n = \dfrac{1}{n^2+1} + \dfrac{2}{n^2+2} + \cdots + \dfrac{n}{n^2+n}$

   (b) $x_n = \left(-\dfrac{1}{2}\right)^n + \sin\left(\dfrac{n\pi}{2}\right)$

   (c) $x_n = \dfrac{n^n + (-n)^n}{2} + \left(1 + \dfrac{1}{2n}\right)^n$

3. Determine whether or not $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly on I, where $u_n(x)$ and I are given in parts (a) and (b) below

   (a) $I = \mathbb{R}$ and $u_n(x) = \begin{cases} 0 & , |x| \leq n \text{ or } |x| \geq n+1 \\ n\sin(1/n^2) & , n < |x| < n+1 \end{cases}$

   (b) $I = [1, \infty)$ and $u_n(x) = \displaystyle\int_1^x e^{-nt^2}\,dt$, $x \in I$.

4. Let D^+ and D^- denote the operation of taking derivatives of real functions from the right and left respectively, for example $D^+f(x) = \lim\limits_{y \to x^+} \dfrac{f(y) - f(x)}{y - x}$, D^- is defined similarly.

   (a) Give an example of a function for which $D^+f(0)$, $D^-f(0)$ both exist but are not equal.

   (b) Prove or disprove: if $D^+f(0)$, $D^-f(0)$ both exist then the function f is continuous at $x = 0$.

5. Suppose that $f(x) = x$ and $g(x) = \begin{cases} 0 & , 0 \leq x < 1/2 \\ 1/2 & , x = 1/2 \\ 1 & , 1/2 < x \leq 1 \end{cases}$, evaluate:

   (a) $\displaystyle\int_0^1 f\,dg$

   (b) $\displaystyle\int_0^1 g\,df$

6. For a nonnegative integer l let $P_l(x) = \sum_{k=0}^{l} a_k x^k$ for real numbers a_k and $x \in [-1, 1]$. Given a positive integer n set $\mathcal{F}(n) = \{P_l(x) \mid 0 \leq l \leq n \text{ and } |a_k| < 1 \text{ for } k = 0, \ldots, l\}$. So $\mathcal{F}(n)$ is

the set of polynomials of degree less than or equal n whose coefficients all have absolute value less than 1. Prove or disprove, for each n the set $\mathcal{F}(n)$ is equicontinuous.

7. Let $f(x,y) = |x|^{1/2}|y|^{1/2} + xy$ be a real function on $\mathbb{R}^2$.

   (a) Find the partial derivatives of f at the origin.

   (b) Discuss the differentiability of f at the origin.

8. Let $x = r\cos(\theta)\sin(\phi)$, $y = r\sin(\theta)\sin(\phi)$, and $z = r\cos(\phi)$ define the map $F(r,\theta,\phi) = (x,y,z)$ from $(r,\theta,\phi) \in \mathbb{R}^3$ to $(x,y,z) \in \mathbb{R}^3$.

   (a) Prove or disprove, F has a global inverse on $\mathbb{R}^3$.

   (b) Find $\dfrac{\partial}{\partial x}\theta(0,1,0)$.