

GRADUATE PRELIMINARY EXAMINATION

ANALYSIS

26 August 1994

✓(1) For which real x does the series $\sum_{n=1}^{\infty} ne^{-nx}$ converge?

✓(2) Suppose that f is a differentiable function on $(0, \infty)$, $\lim_{x \rightarrow \infty} f(x)/x = 0$, and $\lim_{x \rightarrow \infty} f'(x) = a$. Prove that $a = 0$.

✓(3) ^{bounded} Find $\lim_{n \rightarrow \infty} x_n$ when $x_{n+1} = \sqrt{x_n + a}$, $a > 0$, and $x_1 = \sqrt{a}$. 2/3

✓(4) Prove that if a function $f(x)$ is integrable on $[a, b]$ then its absolute value $|f(x)|$ is also integrable on $[a, b]$ and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

✓(5) Let f be a complex valued function on a set D and suppose that $|f(x)| < 1$ for each $x \in D$.

- ✓(a) Show that the sequence of powers of f , $\{f, f^2, f^3, \dots\}$ converges pointwise.
 ✓(b) Find necessary and sufficient conditions for the convergence to be uniform.

✓(6) Let $K(x, y)$ be continuous on the rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$. For integrable functions f on $[c, d]$ define an operator T by

$$(Tf)(x) = \int_c^d K(x, y)f(y)dy.$$

- ✓(a) Show that $(Tf)(x)$ is a continuous function on $[a, b]$.
 ✓(b) Show that $S = \{Tf \mid \int_c^d |f(x)|dx \leq 1\}$ is an equicontinuous family of functions on $[a, b]$.

✓(7) Let $U = \{(u, v) \in \mathbb{R}^2 \mid u > 0\}$ and define $F: U \rightarrow \mathbb{R}^2$ by $F(u, v) = (u \cos v, u \sin v) = (x, y)$.

✓(a) Show that F is an open mapping on U .

→ ✓(b) Find $\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$.

✓(8) Let $f(x, y) = x^2 + y^2 - 5$ be a function on \mathbb{R}^2 .

✓(a) Describe thoroughly the results of applying the implicit function theorem in a neighborhood of the point $(2, 1)$.

✓(b) Describe thoroughly the results of applying the implicit function theorem in a neighborhood of the point $(\sqrt{5}, 0)$.