Graduate Proficiency Examination
Analysis

Fall 1993

Instructions: Do all problems. Each problem is worth 10 points.

1. Given a C^1 function $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\|F(x)\| \leq \|x\|^2, \qquad x \in \mathbb{R}^n,$$

prove that there is an $\epsilon > 0$ such that the equation $F(x) = x + \alpha$ has a solution x whenever the vector α satisfies $\|\alpha\| < \epsilon$.

2. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$ prove that there exists a sequence b_n such that $\lim_{n\to\infty} b_n = +\infty$ *and* $\sum_{n=1}^{\infty} a_n b_n$ converges.

3. Assume that the family $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $[0, 1]$ is equicontinuous and pointwise bounded. Also assume $\int_a^b f_n(x)\,dx \to 0$ as $n \to \infty$ for every $0 \leq a < b \leq 1$. Prove that $f_n \to 0$ uniformly.

4. Let P_E denote the set of real-valued polynomials which involve no odd powers of the variable, i.e., the coefficient of each odd power term is zero. Prove that P_E is dense in $C([0, 1])$ with the sup norm. For which closed intervals other than $[0, 1]$ can the same be proved?

5. For which non-decreasing functions β on $[0, 1]$ does the Riemann-Stieltjes integral $\int_0^1 \beta d\beta$ exist? Prove your assertion.

6. If f is continuous and $\lim_{s\to\infty} f(s) = a$, prove that $\frac{1}{\log t}\int_1^t \frac{f(s)}{s}\,ds \to a$ as $t \to \infty$.