August, 2004

Syracuse University, Department of Mathematics
Sample Algebra Graduate Preliminary Examination

8/5/04

Instructions: This exam contains ten problems. You are not allowed to use your notes, and you should not collaborate with one another.

1. For the indicated values of $c(x)$ and $m(x)$, determine whether there exists a square complex matrix A for which $c(x)$ is the characteristic polynomial and $m(x)$ is the minimal polynomial. If such an A exists, find all possible Jordan normal forms of A. Justify your answers.
(a) $c(x) = (x-2)(x+3)^2(x+4)$ and $m(x) = (x-2)(x+3)$.
(b) $c(x) = (x+1)^2(x-3)^3$ and $m(x) = (x+1)(x-3)^2$.

2. Assume that $\langle\ ,\rangle$ is a nondegenerate bilinear form on a finite-dimensional vector space V. Determine if the following are true or false. If true, prove it; if false, provide a counterexample.
(a) If $T : V \to V$ is a linear operator, then there exists a unique linear operator $T^* : V \to V$ such that $\langle T^*(v), w\rangle = \langle v, T(w)\rangle$ for all $v, w \in V$.
(b) If $W \subseteq V$ is any subspace, then the restriction of $\langle\ ,\rangle$ to W is again nondegenerate.

3. Let $N : V \to V$ be a normal linear operator on a finite-dimensional inner product space V over the field of complex numbers. If v and w are eigenvectors of N corresponding to different eigenvalues, prove that v and w are orthogonal.

4. (a) Find all groups of order 9 up to isomorphism.
(b) Find the automorphism group of each of the groups found in part (a).
(c) Find all groups of order 18 up to isomorphism.
Justify your answers.

5. Find the center of the group $\mathrm{GL}_n(\mathbb{R})$ of $n \times n$ nonsingular real matrices. Justify your answer.

6. Let R be an integral domain.
(a) Define what is a prime element of R.
(b) Define what is an irreducible element of R.
(c) Show that prime implies irreducible, or give an example to show that it is not true.
(d) Show that irreducible implies prime, or give an example to show that it is not true.

7. Denote by A the factor group of the free abelian group with a free basis $\{x, y, z\}$ modulo the subgroup generated by the elements $3x + 2y + 8z$ and $2x + 4z$. Write A as a direct sum of indecomposable cyclic subgroups.

8. Let $F \subseteq E$ be an extension of fields.
(a) Define what it means for this to be a finite extension.
(b) Define what it means for this to be an algebraic extension.

(c) Show that a finite extension is algebraic.

(d) If K is an intermediate field, $F \subseteq K \subseteq E$, with $F \subseteq K$ and $K \subseteq E$ both algebraic, show that $F \subseteq E$ is again algebraic.

9. Let F be a field of characteristic 0, and let E be a splitting field for an irreducible polynomial $f \in F[X]$. Let K be an intermediate field, $F \subseteq K \subseteq E$, and assume that $F \subseteq K$ is a Galois extension. If g_1, g_2 are irreducible factors of $f \in K[X]$, show that there exists an F-automorphism σ of E such that $\hat{\sigma}(g_1) = g_2$. (Here $\hat{\sigma}$ is the natural extension of σ to $E[X]$.)

10. If $F \subseteq E$ is a finite extension with finitely many intermediate fields, show that $F \subseteq E$ is a simple extension, that is $E = F[\gamma]$ for some $\gamma \in E$.