

Ph. D. Algebra Preliminary Exam Wednesday, January 10, 2007

Show the work you do to obtain an answer. Give reasons for your answers. There are 10 questions. Do all parts of all questions.

Notations: $C(x)$ is the centralizer of a group element x , $\langle \cdot, \cdot \rangle$ is the usual inner product and $\| \cdot \|$ is the usual metric on \mathbb{R}^n , A^T is the transpose of a matrix A , I_n is the $n \times n$ identity matrix.

1. (10 points) Let G be a finite group having exactly one nontrivial proper subgroup. Prove that G is cyclic of order p^2 for some prime number p .

2. Let G be a finite group having n distinct conjugacy classes.

(a) (8 points) Prove the identity $\sum_{x \in G} |C(x)| = n|G|$.

(b) (2 points) Compute the probability that two randomly chosen elements of G commute. The random selection is done "with replacement" so that choosing the same element twice is a possible outcome.

3. (10 points) Let A be an $n \times n$ real matrix, and prove that the following (criteria for A to be orthogonal) are equivalent.

(a) $\|AX\| = \|X\|$ for all $X \in \mathbb{R}^n$;

(b) $\langle AX, AY \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathbb{R}^n$;

(c) $A^T A = I_n$.

4. (10 points) Let G be a simple group of order 60. Prove that G has 10 Sylow 3-subgroups. (Do not use the fact that $G \cong A_5$.)

5. Let S_5 denote the symmetric group on five elements.

(a) (5 points) Find a representative for each conjugacy class of S_5 , and determine the number of elements in each class.

(b) (5 points) Find all elements of S_5 that commute with the 4-cycle (1234). Justify your answer.

6. Let \mathbb{Q} be the rational numbers.

(a) (8 points) Find the minimal polynomial for $\sqrt{6} + \sqrt{10}$ over \mathbb{Q} . Be sure to prove that it is the minimal polynomial.

(b) (2 points) What is the degree of the field extension $\mathbb{Q} \subset \mathbb{Q}[\sqrt{6} + \sqrt{10}]$?

7. (10 points) Let $F \subset K \subset L$ be a tower of field extensions. Provide either a proof or a counter example for the following statement. If $F \subset K$ and $K \subset L$ are both Galois extensions then $F \subset L$ is a Galois extension.

8. (10 points) Let R be a commutative ring with identity and let I and J be ideals of R . Prove: if $I + J = (1)$ then $IJ = I \cap J$.

9. Let A be a square matrix over the complex numbers. Assume that the minimal polynomial of A is $(x - 2)^2$ and the characteristic polynomial of A is $(x - 2)^5$.

(a) (6 points) Give all the possible Jordan canonical forms for such an A .

(b) (4 points) For each of the possibilities in (a) compute the nullity of $(A - 2I)^k$ for all positive integers k . (I is the identity matrix of the same size as A .)

10. (10 points) An Abelian group is generated by $w, x, y,$ and z subject to the relations: $w + 3x + 3y + 5z = 0$, $w + x + y = 0$, $2x + 2y + 2z = 0$, and $3z = 0$. Express what group this is in each of the two ways that appear in the structure theorem for finitely generated Abelian groups.