Department of Mathematics – Syracuse University
Graduate Algebra Preliminary Examination January 27th, 2006

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Name: ______________________________________

This exam contains 10 problems. You are not allowed to use notes or to collaborate with one another.

1. Assume A is an n-by-n matrix such that $rk[A-2I_n]-rk[(A-2I_n)^2]=5$. (Here rk denotes the rank of the matrix and I_n is the identity matrix.) How much can be concluded about the Jordan canonical form of A?

2. Let $V = R^2$ be the Euclidean plane and assume that $T : V \to V$ is a linear operator. Let l_1, l_2 and l_3 be three distinct lines passing through the origin with $T(l_i) = l_i$, for $i = 1,2,3$. Show that T is a dilation, that is, T is multiplication by some constant.

3. Let k be a field and let $T : V \to W$ be a linear transformation between two vector spaces over k.

(a) (2 points.) Define the transpose $T^* : W^* \to V^*$ of the linear operator T

(b) (7 Points.) Show that T^* is injective if and only if T is onto.

4. Show that a finite group of order 24 cannot be simple.

5. Let S_n denote the symmetric group on n letters and let A_n denote the alternating subgroup. Recall that if $\sigma \in G$, where G is a group, the centralizer of σ in G is the subgroup $C_G(\sigma) = \{\tau \in G \mid \tau\sigma = \sigma\tau\}$.

(a) (3 Points) If $\sigma \in A_n$, use the sign homomorphism from S_n to $\{\pm 1\}$, to show that $C_{A_n}(\sigma)$, the centralizer of σ in A_n, is either equal to $C_{S_n}(\sigma)$ or it is a subgroup of $C_{S_n}(\sigma)$ of index 2.

(b) (3 Points) If $n = 5$ and $\sigma \in A_n$ is a 3-cycle, show that $[C_{S_n}(\sigma) : C_{A_n}(\sigma)] = 2$.

(c) (4 Points) We know that all 3-cycles are conjugate in S_5. Use this and part (b) to show that all 3-cycles are conjugate in A_5.

6. Let G be a finite p-group for some prime p. Show that the center of G is not trivial.

7. Let Q denote the field of rational numbers and let $f = X^3 + 2X^2 + 7 \in Q[X]$.

(a) (3 Points.) Show that f has precisely one real root.

(b) (3 Points.) Show that f is irreducible in $Q[X]$.

(c) (4 Points.) Show the Galois group of f over Q is isomorphic to the symmetric group S_3.

8. (a) (5 Points.) Let K/F be a finite extension of fields. Show that K/F is algebraic.

(b) (5 Points.) Let L/K and K/F be algebraic field extensions. Show that L/F is also algebraic.

9. (a) (3 Points) Assume R is a commutative ring and $I \subseteq R$ is an ideal. Show that $I[X] \subseteq R[X]$ is an ideal.

(b) (7 Points.) Using the first isomorphism theorem or otherwise, show that $R[X]/I[X]$ is isomorphic to $(R/I)[X]$

10. (a) (2 Points.) Let S be a commutative ring and assume that $I = Sf + Sg$ is the ideal generated by two elements f and g. Show that if $h \in S$ is any element, then I is also generated by the elements f and $g - hf$.

(b) (4 Points.) Let Z be the ring of integers and assume I is an ideal of $Z[X]$ generated by the set $f, g \in Z[X]$. Show that we can replace f and g by two generators, one of which has a zero constant term.