

ALGEBRA

1. Let A be a complex matrix with characteristic polynomial $(x-1)^4(x+2)^3$. Assume that the rank of $(A-I_7)^2$ is 5 and the rank of $(A+2I_7)$ is 4. What is the Jordan canonical form of A ? Justify your answer.

2. (a) Let A be an n -by- n matrix over a field F . Then $C(A) = \{X \in M_n(F) \mid XA = AX\}$ is called the centralizer of A in $M_n(F)$. Let $Y \in M_n(F)$ be an invertible matrix. Show that $C(YAY^{-1}) = Y[C(A)]Y^{-1}$. (Note: $Y[C(A)]Y^{-1} = \{YXY^{-1} \mid X \in C(A)\}$.)
 (b) If F is the field complex numbers and $n = 2$, what is the smallest dimension $C(A)$ can have?

3. For each of the following statements, either prove it or give an example to show that it is false.
 - (a) Assume $\phi: V \rightarrow W$ is a linear transformation between vector spaces. If $\{v_1, v_2, \dots, v_n\}$ is a subset for V with $\{\phi(v_1), \phi(v_2), \dots, \phi(v_n)\}$ linearly independent in W , then $\{v_1, v_2, \dots, v_n\}$ is linearly independent in V .
 - (b) Assume V is a 5-dimensional vector space and W is a 3 dimension vector space with $X: V \rightarrow W$ and $Y: V \rightarrow W$ surjective linear transformations. Then there exists $v \in V$, nonzero, such that $X(v) = Y(v) = 0$.

4. Show that an n -by- n matrix A over a field F is similar to a diagonal matrix if and only if there is a basis for $F^{(n)}$, the space of n -by- 1 matrices over F , consisting of eigenvectors for A .

5. Let P_2 be the vector space of polynomials of degree at most 2 over the real numbers together with the inner product $\langle f \mid g \rangle = \int_0^2 fg \, dx$. Let $\phi: P_2 \rightarrow \mathbb{R}$ be the functional given by $\phi(f) = f(1)$. Find $g \in P_2$, such that $\phi(f) = \langle f \mid g \rangle$, for all $g \in P_2$.

6. Let V be a finite dimensional inner product space over the complex numbers and let W be a subspace with orthonormal basis $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$. If $\beta \in V$, show that $\gamma = \sum_i \langle \beta \mid \alpha_i \rangle \alpha_i$ is the unique element of W with $\|\beta - \gamma\| = \sqrt{\langle \beta - \gamma \mid \beta - \gamma \rangle}$ minimal.