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Syracuse University, Department of Mathematics
Algebra Graduate Preliminary Examination

Instructions: This exam contains ten problems. You are not allowed to use your notes, and you should not collaborate with one another.

1. Find all possible Jordan normal forms of a complex square matrix A with the characteristic polynomial $x(x+1)^3(x-3)^2$ if $A+I$ has rank 4.

2. Let f be a bilinear form on a finite-dimensional vector space V over a field k, and let B be the matrix of f with respect to some basis for V. For each $\alpha \in V$ define a function $\phi_\alpha : V \to k$ by $\phi_\alpha(\beta) = f(\alpha, \beta)$.

(a) Prove that ϕ_α is a functional on V.

(b) If $\hat{V}$ is the dual space of V, prove that the map $\sigma : V \to \hat{V}$ given by $\sigma(\alpha) = \phi_\alpha$ is a linear transformation.

(c) Find and prove the necessary and sufficient conditions on B in order for σ to be an isomorphism.

3. Let l_1, l_2, l_3 be three distinct straight lines passing through the origin of the Euclidean plane $\mathbb{R}^2$. Prove that if m_1, m_2, m_3 are three distinct lines through the origin, then there exists a linear automorphism ϕ of $\mathbb{R}^2$ satisfying $\phi(l_i) = m_i,\ i = 1, 2, 3$.

4. Let $GL_n(F)$ denote the group of $n \times n$ nonsingular matrices over a field F.

(a) Prove that the map sending a complex number $a + bi$ to the 2×2 real matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a homomorphism of the field of complex numbers into the ring of 2×2 real matrices.

(b) Find a subgroup of $GL_2(\mathbb{R})$ isomorphic to the multiplicative group of nonzero complex numbers.

(c) Prove that for every n, $GL_n(\mathbb{C})$ is isomorphic to a subgroup of $GL_{2n}(\mathbb{R})$.

5. Give a complete list of nonisomorphic groups of order 245, and prove your answer.

6. Let $\mathbb{Z}[X]$ be the ring of polynomials in the variable X with integer coefficients. Determine whether the following statements are true or false. If a statement is true, give a proof; if it is false, provide a counterexample.

(a) $\mathbb{Z}[X]$ is an integral domain.

(b) $\mathbb{Z}[X]$ is a principal ideal domain.

(c) $\mathbb{Z}[X]$ is a unique factorization domain.

(d) Let $\mathbb{Z}^3$ be the set of triples of integers. Given a matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, we turn $\mathbb{Z}^3$ into a $\mathbb{Z}[X]$-module by putting $p(X) \cdot \mathbf{v} = p(A)\mathbf{v}$ for all $p(X) \in \mathbb{Z}^3$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then $\mathbb{Z}^3$ is a torsion $\mathbb{Z}[X]$-module.

7. Let R be a principal ideal domain with the field of fractions K. If $M \subseteq K$ is a finitely generated R-submodule of K, show that M is generated by one element.

8. Let α be the real cube root of 2. Compute the irreducible polynomial for $1 + \alpha^2$ over $\mathbb{Q}$.

9. Let $K = F(\alpha)$ be a field extension generated by an element α, and let $\beta \in K, \beta \notin F$. Prove that α is algebraic over the field $F(\beta)$.

10. Let $K \supset L \supset F$ be fields of characteristic 0. Prove or disprove:

(a) If K/F is Galois, then K/L is Galois.

(b) If K/F is Galois, then L/F is Galois.

(c) If L/F and K/L are Galois, then K/F is Galois.