Preliminary Exam - January 2003

1. A 5-by-5 matrix A has characteristic polynomial $(x-2)^3(x+1)^2$, while the matrix $(A-2I_5)^2$ has rank 2 and $A+I_5$ has rank 4. What are the possible Jordan canonical forms of A?

2. If A is a Hermitian complex matrix, show that its characteristic values must be real. [Recall that A is called Hermitian (or self adjoint) if it satisfies the equation $A = \overline{A^t}$, where $\overline{A^t}$ is the complex conjugate of the transpose of A.]

3. Let V be a vector space with basis $B = \{v_1, v_2, ..., v_n\}$ and let $w \in V$ be nonzero. Show directly, without quoting the dimension theorem, that we can find i such that we can replace v_i in B by w and still have a basis for V.

4. Let V, (|) be a finite dimensional inner product space over the real numbers. If W is a subspace of V, prove that we can write V as a direct sum $V = W \oplus W^{\perp}$, where $W^{\perp} = \{v \in V \mid (v \mid w) = 0 \text{ for all } w \in W\}$.

5. Let V and W be finite dimensional vector spaces over a field k and let $T: V \to W$ be a linear transformation.

   (a) Define the transpose map $T^* : W^* \to V^*$ where $W^* = Hom_k(W,k)$ is the dual of W.

   (b) Show that T^* is injective if and only if T is surjective.

6. Let V and W be finite dimensional vector spaces over a field k and let $T: V \to W$ be a linear transformation. Let $S = \{v_1, v_2, ..., v_m\}$ be a subset of V. For each of the following statements either prove it or give a counter example to it.

   (a) If S is linearly independent set in V, then $\{T(v_1), T(v_2), ..., T(v_m)\}$ must be a linearly independent set in W.

   (b) If $\{T(v_1), T(v_2), ..., T(v_m)\}$ is a linearly independent set in W, then S must be a linearly independent set in V.