PhD Preliminary examination, August 2002

(2 hours)

You may do these problems in any order you want. Start each problem on a new page.

1. Prove or disprove the following: If A is a *complex* square matrix such that $A^n = A$ for some integer $n > 1$, then A is diagonalizable.

2. Let V be the vector space of all the real polynomials of degree less or equal to 3, and let $T: V \to V$ be the linear transformation given by $T(f) = -f + f' + f'''$.
(a) Find the matrix M of T with respect to the basis $\{1, x, x^2, x^3\}$ of V.
(b) Find the minimal polynomial of T.
(c) Is the matrix M diagonalizable ? Why, or why not?
(d) Find the Jordan canonical form of M.

3. Let A be a fixed 5×8 real matrix for which there exists an 8×5 real matrix B satisfying $AB = I$ where I is the identity matrix.
(a) Prove that B can be chosen in such a way that three of its rows consist entirely of zeros.
(b) What are the necessary and sufficient conditions on the matrix A for the uniqueness of the matrix B satisfying (a).
(c) Suppose that now B is a fixed 8×5 matrix, and A varies. State, but do not prove the analogue of (a).

4. Let V be the vector space of all the real polynomials of degree less or equal to 3. For all $p(x) \in V$, put $\varphi(p(x)) = \int_1^2 p(x)dx$.
(a) Prove that φ is a linear functional on V.
(b) Let $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ in V^* be the dual basis of the basis $\{1, x, x^2, x^3\}$ of V. Express φ as a linear combination of the φ_i.
(c) Give the definition of the evaluation map $e: V \to V^{**}$.
(d) Find $e(1 + x + x^2 + x^3)(\varphi)$ where φ is defined above.
(e) Show that e is a monomorphism. Is it an isomorphism ? Why, or why not?

5. (a) Show that the eigenvalues of a real symmetric matrix are real.
(b) Let A be a real matrix. Show that $A'A$ is diagonalizable.

6. For $A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{bmatrix}$, find a real orthogonal matrix P and a diagonal matrix D such that $A = PDP'$. Hint: 6 is one of the eigenvalues.