Ph. D. Preliminary Examination in Algebra, August 21, 1999

Show your work: **no work, no credit. No calculators are allowed.**

1. Let $\mathbf{P}_4$ be the vector space of real polynomials of degree ≤ 4 in the indeterminate x. For $a \in \mathbf{R}$, we put $\mathbf{P}_4(a) = \{f \in \mathbf{P}_4 | f(a) = 0\}$.

(a) Prove $\mathbf{P}_4(a)$ is a subspace of $\mathbf{P}_4$.

(b) Find a basis for and the dimension of $\mathbf{P}_4(a)$.

(c) Find a basis for and the dimension of $\mathbf{P}_4(-3) \cap \mathbf{P}_4(2)$.

2. For the indicated values of $c(x)$ and $m(x)$, determine whether there exists a square complex matrix A for which $c(x)$ is the characteristic polynomial and $m(x)$ is the minimal polynomial. If such an A exists, find all possible Jordan normal forms of A. Justify your answers.

(a) $c(x) = x(x+1)(x-2)^3$ and $m(x) = x(x-2)^2$.

(b) $c(x) = (x-4)^2(x+3)^3$ and $m(x) = (x-4)(x+3)^2$.

3. Let A be a 4×3 matrix of rank 3 over a field F.

(a) Is there a matrix B satisfying $BA = I_3$, where I_3 is the 3×3 identity matrix?

(b) Let $T_A : F^3 \to F^4$ be the linear transformation given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in F^3$. Is T_A one-to-one? Is it onto?

Justify your answers.

4. Let $\phi : U \to V$ be a linear transformation of finite-dimensional vector spaces U, V over a field F, and let $\hat{\phi} : \hat{V} \to \hat{U}$ be the dual linear transformation. Prove that ϕ is onto if and only if $\hat{\phi}$ is one-to-one.

5. Let U and V be subspaces of the Euclidean space $\mathbf{R}^n$. If $\dim U < \dim V$, prove that there is a non-zero vector in V orthogonal to all vectors in U.

6. Give an example of a normal linear operator on a finite-dimensional unitary space that is neither self-adjoint nor unitary. Justify your answer.