Ph.D Preliminary Examination. Fall 1998. Algebra.

1. Let V be a finite dimensional vector space. Prove that the dimension of V is even if and only if there is a linear map $f: V \to V$ such that $Ker f = \operatorname{Im} f$.

2. Let V be a finite dimensional complex vector space and let $\phi: V \to V$ be a linear map.
(a) Assume that for each natural number k, $trace(\phi^k) = 0$. Prove that 0 is an eigenvalue of ϕ.
(b)Prove that ϕ is nilpotent if and only if for each natural number k, $trace(\phi^k) = 0$.

3. Find two matrices having the same rank and the same characteristic polynomial, but not similar to each other.

4. Let A and B be two self - adjoint matrices. Show that AB is self - adjoint if and only if $AB = BA$.

5. Let V be a n-dimensional real vector space, and let q be a quadratic form on V. Let $A = (a_{ij})_{1 \le i,j \le n}$ be the symmetric matrix of q in an ordered basis. Show that if the form q is positive definite, then for each positive integer k, we have $\det A_k > 0$, where $A_k = (a_{ij})_{1 \le i,j \le k}$

6. (a) Show that every $n \times n$ matrix A can be uniquely written as the sum of a symmetric and a skew-symmetric matrix.
(b) Let A and B be two congruent $n \times n$ matrices. Show that A' and B' are also congruent.
(c) Again, let A and B be two congruent $n \times n$ matrices, and write $A = A_1 + A_2$ and $B = B_1 + B_2$, where A_1 and B_1 are symmetric and A_2 and B_2 are skew-symmetric. Show that A_1 is congruent to B_1, and that A_2 is congruent to B_2.