Written Ph. D. Qualifying Examination in Linear Algebra (MAT 635)

Fall 1993

1. Let A be a real symmetric matrix satisfying $A^k = I$ for some positive integer k, where I is the identity matrix of the same size as A. Prove $A^2 = I$.

2. Let v be a nonzero vector of the Euclidean space $\mathbb{R}^n$. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator given by the formula $T(x) = x - 2\langle x, v\rangle v$ for all $x \in \mathbb{R}^n$, where $\langle\ ,\ \rangle$ is the standard inner product. Prove that T can be represented by the matrix

$$\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix},$$

where I is the $(n-1)\times(n-1)$ identity matrix.

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator represented by the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

with respect to the standard basis. Show there exist nonzero T-invariant subspaces U and V of $\mathbb{R}^3$ satisfying $\mathbb{R}^3 = U \oplus V$.

4. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation of rank k. Show there exist linear transformations $U: \mathbb{R}^n \to \mathbb{R}^k$ and $V: \mathbb{R}^k \to \mathbb{R}^m$, where U is onto and V is one-to-one, satisfying $T = VU$.

5. Denote by $\mathrm{Mat}_n(\mathbb{R})$ the set of all real $n \times n$ matrices. A matrix $N \in \mathrm{Mat}_n(\mathbb{R})$ is called nilpotent if $N^k = 0$ for some positive integer k.

(a) Do all nilpotent matrices form a subspace of $\mathrm{Mat}_n(\mathbb{R})$?

(b) Prove $I+N$ is invertible, where $I \in \mathrm{Mat}_n(\mathbb{R})$ is the identity matrix.

(c) Show $I+N$ is diagonalizable if and only if $N=0$.

6. Let $A=[a_{ij}]$ be the $n \times n$ real matrix satisfying $a_{ij}=1$ for all $i, j=1, \ldots, n$. Denote by the same letter A the linear operator $\mathbb{R}^n \to \mathbb{R}^n$ whose representation matrix with respect to the standard basis is A.

(a) Describe Ker A and Im A as subsets of $\mathbb{R}^n$.

(b) What is the minimal polynomial of A?

(c) Show A is diagonalizable.